Discrete Math Trees Quiz Solutions

1. We are given a tree with 10 vertices is a connected acyclic graph with 9 edges.

    a. **If you remove any edge, the resulting graph will be disconnected.**
    True - Every edge in a tree is a bridge. Removing any edge will disconnect the tree into two components.

    b. **T has a unique path between every pair of vertices.**
    True - by the definition of a tree. Since there are no cycles, there is exactly one path between any two vertices.

    c. **T has exactly two vertices of degree one.**
    False - While a tree must have at least two vertices of degree one (leaves), the exact number can vary. For example, a star-shaped tree with 10 vertices has 9 leaves.

    d. **If you add a new edge, the resulting graph will have a cycle.**
    True - Adding an edge between any two vertices in a tree creates a cycle because there is already a unique path between those vertices.

    Thus, the correct statements are the first, second, and fourth.

2. By the *One More Vertex Than Edges Proposition*, our tree with 42 vertices has **41 edges**.

3. Statement to prove: Every tree (a connected acyclic graph) is bipartite.

    That is, its vertices can be partitioned into two sets A and B such that every edge goes between A and B. To prove by minimal criminal (minimal counterexample) approach we first assume the contrary, let G be a counterexample with the smallest possible number of vertices. Thus G is a tree (connected, acyclic). It is *not* bipartite, and every tree with fewer vertices *is* bipartite.

    A tree with at least 2 vertices has at least 2 leaves (vertices of degree 1), since in a finite tree, any longest path ends in two leaves. Choose a leaf: Let v be a leaf of G, and let u be its unique neighbor. Now consider G′ = G − v, the graph obtained by deleting v and its incident edge.

    G′ is still connected? Yes, because removing a leaf from a connected graph leaves it connected (since v was a leaf, the only path through v went to u, and now it's gone but u is still connected to the rest).

    G′ is still acyclic: removing a vertex cannot create a cycle. So G′ is a tree.

And G' has $n-1$ vertices, where n is the number of vertices in G. By minimality of G (since G' has fewer vertices), G' must be bipartite. So G' has a bipartition into two independent sets (X,Y).

Try to extend the bipartition to G. We want to put v into one of X or Y so that all edges (just uv in G connecting v back) are between X and Y.

There are two possibilities for u:

Case 1: $u \in X$. Then put v in Y. Then uv goes between X and Y. Since v only connects to u, no other edges in G violate bipartiteness. So G is bipartite: contradiction to G being non-bipartite.

Case 2: $u \in Y$. Then put v in X. Same reasoning: uv connects Y to X, and no other edges in G violate bipartiteness. So G is bipartite: contradiction again.

In both cases, G is bipartite. This contradicts our assumption that G is a non-bipartite tree. So no minimal counterexample exists.

Thus every tree is bipartite.